演習問題　2.1

(1) $u = v = w = 0$ を式(2P.1)に代入すると

$$\frac{\partial p}{\partial z} = -\rho g$$

(2) 式(2P.2)にガウスの発散定理を適用して，上式を代入すると

$$F_z = -\iint_{\partial S} p n_z \, dS = -\iiint_V \frac{\partial p}{\partial z} dV = \rho g \iiint_{\mathrm{V}} dV = \rho g V$$

ρV は V に含まれる流体の質量を表し，重力加速度 g をかけると V に含まれる流体の重量を表すことから，アルキメデスの原理が示された．

演習問題　2.2

$\partial/\partial y = \partial/\partial z = 0$ より，式(2.4.9)は

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0 \qquad \cdots (2\mathrm{P}.3)$$

演習問題　2.3

$\partial/\partial y = \partial/\partial z = 0$ より，式(2.4.4)は

$$\frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 + p) + \frac{\partial}{\partial y}(\rho u v) + \frac{\partial}{\partial z}(\rho u w) = \frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 + p) = 0$$

これより

$$\rho\frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t} + u\frac{\partial \rho u}{\partial x} + \rho u\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x}$$

$$= \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x}\right) + u\left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x}\right)$$

$$= \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x}\right) + u\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x}\right) = 0$$

上式に式(2P.3)を代入すると

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \qquad \cdots (2\mathrm{P}.4)$$

演習問題　2.4

$\partial/\partial y = \partial/\partial z = 0$ より，式(2.4.8)は

$$\frac{\partial e}{\partial t} + \frac{\partial}{\partial x}(e+p)u = 0$$

$v = w = 0$ のとき，全エネルギー e は式(2.3.1)より

$$e = \rho\epsilon + \frac{\rho u^2}{2} = \frac{p}{\gamma - 1} + \frac{\rho u^2}{2}$$

ただし，式(3.3.7)を用いた．これより

$$\frac{\partial e}{\partial t} + \frac{\partial}{\partial x}(e+p)u$$

$$= \frac{1}{\gamma-1}\frac{\partial p}{\partial t} + \rho u\frac{\partial u}{\partial t} + \frac{u^2}{2}\frac{\partial \rho}{\partial t} + \frac{\gamma}{\gamma-1}\frac{\partial pu}{\partial x} + \frac{u^2}{2}\frac{\partial \rho u}{\partial x} + \frac{\rho u}{2}\frac{\partial u^2}{\partial x} = 0$$

式(2P.3)および式(2P.4)を用いて ρ と u の時間微分項を消去すると

$$\frac{1}{\gamma-1}\frac{\partial p}{\partial t} + \rho u\frac{\partial u}{\partial t} + \frac{u^2}{2}\frac{\partial \rho}{\partial t} + \frac{\gamma}{\gamma-1}\frac{\partial pu}{\partial x} + \frac{u^2}{2}\frac{\partial \rho u}{\partial x} + \frac{\rho u}{2}\frac{\partial u^2}{\partial x}$$

$$= \frac{1}{\gamma-1}\frac{\partial p}{\partial t} - \rho u\left(u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x}\right) - \frac{u^2}{2}\left(u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x}\right)$$

$$+ \frac{\gamma}{\gamma-1}\frac{\partial pu}{\partial x} + \frac{u^2}{2}\frac{\partial \rho u}{\partial x} + \frac{\rho u}{2}\frac{\partial u^2}{\partial x}$$

$$= \frac{1}{\gamma-1}\frac{\partial p}{\partial t} - u\frac{\partial p}{\partial x} + \frac{\gamma p}{\gamma-1}\frac{\partial u}{\partial x} + \frac{\gamma u}{\gamma-1}\frac{\partial p}{\partial x} = 0$$

整理すると

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \gamma p\frac{\partial u}{\partial x} = 0 \qquad \cdots (2\mathrm{P}.5)$$

演習問題　2.5

式(2P.3)，式(2P.4)，式(2P.5)をまとめると

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho \\ u \\ p\end{pmatrix} + \begin{pmatrix}u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \gamma p & u\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}\rho \\ u \\ p\end{pmatrix} = 0$$

これより

$$A = \begin{pmatrix}u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \gamma p & u\end{pmatrix}$$

Aの固有値λを求めると

$$\begin{vmatrix} u-\lambda & \rho & 0 \\ 0 & u-\lambda & 1/\rho \\ 0 & \gamma p & u-\lambda \end{vmatrix} = (u-\lambda)\left\{(u-\lambda)^2 - \frac{\gamma p}{\rho}\right\} = (u-\lambda)\{(u-\lambda)^2 - a^2\} = 0$$

$$\therefore \quad \lambda = u, \ \lambda = u + a, \ \lambda = u - a$$